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Hidden sl_2-algebra of finite-difference equations[1]

Yuri SMIRNOV[2] and Alexander TURBINER[3]

The connection between polynomial solutions of finite-difference equations and finite-dimensional representations of the sl_2-algebra is established.

Recently it was found [1] that polynomial solutions of differential equations are connected to finite-dimensional representations of the algebra sl_2 of first-order differential operators. In this Talk it will be shown that there also exists a connection between polynomial solutions of finite-difference equations (like Hahn, Charlier and Meixner polynomials) and unusual finite-dimensional representations of the algebra sl_2 of finite-difference operators. So, sl_2-algebra is the hidden algebra of finite-difference equations with polynomial solutions.

First of all, we recall the fact that Heisenberg algebra

$$[a,b] \equiv ab - ba \ = \ 1 \tag{1}$$

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[2]Instituto de Física, UNAM, A.P. 20-364, 01000 México D.F.
On leave of absence from the Nuclear Physics Institute, Moscow State University, Moscow 119899, Russia
E-mail: smirnov@sysul2.ifisicacu.unam.mx
[3]Instituto de Ciencias Nucleares, UNAM, A.P. 70-543, 04510 México D.F.
On leave of absence from the Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: turbiner@nuclecu.unam.mx or turbiner@axcrnb.cern.ch

possesses several representations in terms of differential operators. There is the standard, coordinate-momentum representation

$$a \ = \ \frac{d}{dx} \ , \ b \ = \ x \tag{2}$$

and recently another one was found [2]

$$\begin{aligned} a \ &= \ \mathcal{D}_+ \ , \\ b \ &= \ x(1-\delta\mathcal{D}_-) \ , \end{aligned} \tag{3}$$

where the $\mathcal{D}_\pm$ are translationally-covariant, finite-difference operators

$$\mathcal{D}_+ f(x) \ = \ \frac{f(x+\delta)-f(x)}{\delta} \equiv \frac{(e^{\delta\frac{d}{dx}}-1)}{\delta} f(x) \tag{4}$$

and

$$\mathcal{D}_- f(x) \ = \ \frac{f(x)-f(x-\delta)}{\delta} \equiv \frac{(1-e^{-\delta\frac{d}{dx}})}{\delta} f(x) \ , \tag{5}$$

where δ is a parameter and $\mathcal{D}_+ \to \mathcal{D}_-$, if $\delta \to -\delta$.

Now let us consider the Fock space over the operators a and b with a vacuum $|0>$:

$$a|0> = \ 0 \tag{6}$$

and define an operator spectral problem

$$L[a,b]\varphi(b) = \lambda\varphi(b) \tag{7}$$

where $L[\alpha,\beta]$ is a certain holomorphic function of the variables α,β. We will restrict ourselves studying the operators $L[a,b]$ with polynomial eigenfunctions.

In [1] it was proven that L has a certain number of polynomial eigenfunctions if and only if, L is the sum of two terms: an element of the universal enveloping algebra of the sl_2-algebra taken in the finite-dimensional irreducible representation

$$J_n^+ = b^2a - nb, \ J_n^0 = ba - \frac{n}{2}, \ J_n^- = a \tag{8}$$

where n is a non-negative integer [4] , and an annihilator $B(b)a^{n+1}$, where $B(b)$ is any operator function of b. The dimension of the representation (8) is equal to $(n+1)$ and $(n+1)$ eigenfunctions of L have the form of a polynomial of degree not higher then n. These operators L are named *quasi-exactly-solvable*. Moreover, if L is presented as a finite-degree polynomial in the generators $J^0 \equiv J_0^0$ and $J^- \equiv J_0^-$ only, one can prove that L possesses infinitely-many polynomial eigenfunctions. Such operators L are named *exactly-solvable*.

It is evident that once the problem (7) is solved the eigenvalues will have no dependence on the particular representation of the operators a and b. This allows us to construct isospectral operators by simply taking different representations of the operators a and b in the problem (7). In particular, this implies that if we take the representation (3), then the eigenvalues of the problem (7) do not depend on the parameter δ !

Without loss of generality one can choose the vacuum

$$|0> = \ 1 \tag{9}$$

and then it is easy to see that [2]

$$b^n|0>= [x(1-\delta\mathcal{D}_-)]^n|0>= \ x(x-\delta)(x-2\delta)\dots(x-(n-1)\delta) \equiv x^{(n)} \ . \tag{10}$$

This relation leads to a very important conclusion: Once a solution of (7) with a, b (2) is found,

$$\varphi(x) = \sum \alpha_k x^k \ , \tag{11}$$

then

$$\tilde{\varphi}(x) = \sum \alpha_k x^{(k)} \tag{12}$$

is the solution of (7) with a, b given by (3).

Now let us proceed to a study of the second-order finite-difference equations with polynomial solutions and find the corresponding isospectral differential equations.

The standard second-order finite-difference equation relates an unknown function at three points and has the form[4]

$$A(x)\varphi(x+\delta) - B(x)\varphi(x) + C(x)\varphi(x-\delta) = \lambda\varphi(x), \tag{13}$$

[4]Taking a, b from (2), the algebra (8) becomes the well-known realization of sl_2 in first-order differential operators. If a, b from (3) are chosen then (8) becomes a realization of sl_2 in finite-difference operators.

where $A(x), B(x), C(x)$ are arbitrary functions, $x \in R$. One can pose a natural problem: *What are the most general coefficient functions $A(x), B(x)$, $C(x)$ for which the equation (13) admits infinitely-many polynomial eigenfunctions ?* Basically, the answer is presented in [1]. Any operator with the above property can be represented as a polynomial in the generators J^0, J^- of the sl_2-algebra:

$$J^+ = x(\frac{x}{\delta} - 1)e^{-\delta \frac{d}{dx}}(1 - e^{-\delta \frac{d}{dx}}) \ ,$$

$$J^0 = \frac{x}{\delta}(1 - e^{-\delta \frac{d}{dx}}) \ , \ J^- = \frac{1}{\delta}(e^{\delta \frac{d}{dx}} - 1) \ . \tag{14}$$

which is the hidden algebra of our problem. One can show that the most general polynomial in the generators (14) leading to (13) is

$$\tilde{E} = A_1 J^0 J^0 (J^- + \frac{1}{\delta}) + A_2 J^0 J^- + A_3 J^0 + A_4 J^- + A_5 \ , \tag{15}$$

and in explicit form,

$$\begin{gathered}[\frac{A_4}{\delta} + \frac{A_2}{\delta^2}x + \frac{A_1}{\delta^3}x^2]e^{\delta \frac{d}{dx}} + \\ [A_5 - \frac{A_4}{\delta} + (\frac{A_1}{\delta^2} - 2\frac{A_2}{\delta^2} + \frac{A_3}{\delta})x - 2\frac{A_1}{\delta^3}x^2] + \\ [-(\frac{A_1}{\delta^2} - \frac{A_2}{\delta^2} + \frac{A_3}{\delta})x + \frac{A_1}{\delta^3}x^2]e^{-\delta \frac{d}{dx}}\end{gathered} \tag{16}$$

where the A's are free parameters. The spectral problem corresponding to the operator (16) is given by

$$\begin{gathered}(\frac{A_4}{\delta} + \frac{A_2}{\delta^2}x + \frac{A_1}{\delta^3}x^2)f(x+\delta) - \\ [-A_5 + \frac{A_4}{\delta} - (\frac{A_1}{\delta^2} - 2\frac{A_2}{\delta^2} + \frac{A_3}{\delta})x + 2\frac{A_1}{\delta^3}x^2]f(x) + \\ [-(\frac{A_1}{\delta^2} - \frac{A_2}{\delta^2} + \frac{A_3}{\delta})x + \frac{A_1}{\delta^3}x^2]f(x-\delta) = \lambda f(x) \ .\end{gathered} \tag{17}$$

with the eigenvalues $\lambda_k = \frac{A_1 k^2}{\delta} + A_3 k$. This spectral problem has Hahn polynomials $h_k^{(\alpha,\beta)}(x, N)$ of the *discrete* argument $x = 0, 1, 2 \ldots, (N-1)$

as eigenfunctions (we use the notation of [4]). Namely, these polynomials appear, if $\delta = 1, A_5 = 0$ and

$$A_1 = -1,\ A_2 = N - \beta - 2,\ A_3 = -\alpha - \beta - 1,\ A_4 = (\beta + 1)(N - 1)\ .$$

If, however,

$$A_1 = 1,\ A_2 = 2 - 2N - \nu,\ A_3 = 1 - 2N - \mu - \nu,\ A_4 = (N + \nu - 1)(N - 1)$$

the so-called analytically-continued Hanh polynomials $\tilde{h}_k^{(\mu,\nu)}(x, N)$ of the *discrete* argument $x = 0, 1, 2 \ldots, (N - 1)$ appear, where $k = 0, 1, 2 \ldots$.

In general, our spectral problem (17) has Hahn polynomials $h_k^{(\alpha,\beta)}(x, N)$ of the *continuous* argument x as polynomial eigenfunctions. We must emphasize that our Hahn polynomials of the *continuous* argument *do not* coincide to so-called continuous Hahn polynomials known in literature[5].

So Equation (17) corresponds to the most general exactly-solvable finite-difference problem, while the operator (15) is the most general element of the universal enveloping sl_2-algebra leading to (13). Hence the Hahn polynomials are related to the finite-dimensional representations of a certain cubic element of the universal enveloping sl_2-algebra (for a general discussion see [3]).

One can show that if the parameter N is integer, then the higher Hahn polynomials $k \geq N$ have a representation

$$h_k^{(\alpha,\beta)}(x, N) = x^{(N)} p_{k-N}(x)\ , \tag{18}$$

where $p_{k-N}(x)$ is a certain Hahn polynomial. It explains an existence the only a finite number of the Hahn polynomials of *discrete* argument $x = 0, 1, 2 \ldots, (N - 1)$. Similar situation occurs for the analytically-continued Hanh polynomials

$$\tilde{h}_k^{(\mu,\nu)}(x, N) = x^{(N)} \tilde{p}_{k-N}(x)\ ,\ k \geq N\ , \tag{19}$$

where $\tilde{p}_{k-N}(x)$ is a certain analytically-continued Hahn polynomial.

Furthermore, if we take the standard representation (8) for the algebra sl_2 at $n = 0$ with a, b given by (2) and plug it into (15), the third order differential operator *isospectral* to (16)

$$\tilde{E}_2(\frac{d}{dx}, x) = A_1 x^2 \frac{d^3}{dx^3} + [(A_1 + A_2) + \frac{A_1}{\delta} x] x \frac{d^2}{dx^2} + [A_4 + (\frac{A_1}{\delta} + A_3) x] \frac{d}{dx} + A_5 \tag{20}$$

appears, which possesses polynomial eigenfunctions.

Taking in (17) $\delta = 1, A_5 = 0$ and putting

$$A_1 = 0, A_2 = -\mu,\ A_3 = \mu - 1, A_4 = \gamma\mu\ ,$$

and if $x = 0, 1, 2 \ldots, (N-1)$, we reproduce the equation, which has the Meixner polynomials as eigenfunctions. Furthermore, if

$$A_1 = 0, A_2 = 0,\ A_3 = -1, A_4 = \mu\ ,$$

Equation (17) corresponds to the equation with the Charlier polynomials as eigenfunctions (for the definition of the Meixner and Charlier polynomials see, e.g., [4]). For a certain particular choice of the parameters, one can reproduce the equations having Tschebyschov and Krawtchouk polynomials as solutions. If x is the continuous argument, we will arrive at *continuous* analogues of above-mentioned polynomials. Up to our knowledge those polynomials are not studied in literature.

Among the equations (13) there also exist quasi-exactly-solvable equations possessing a finite number of polynomial eigenfunctions. All those equations are classified via the cubic polynomial element of the universal enveloping sl_2-algebra taken in the representation (3), (8)

$$\tilde{T} = A_+(J_n^+ + \delta J_n^0 J_n^0) + A_1 J_n^0 J_n^0 (J_n^- + \frac{1}{\delta}) + A_2 J_n^0 J_n^- + A_3 J_n^0 + A_4 J_n^- + A_5 \quad (21)$$

(cf. (15)), where the A's are free parameters.

In conclusion, it is worth emphasizing a quite surprising result: In general, three-point (quasi)-exactly-solvable *finite-difference* operators of the type (13) emerging from (15), or (21) are isospectral to (quasi)-exactly-solvable, third-order *differential* operators.

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